Box complexes and Kronecker double coverings (New topics of transformation groups)

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1 Introduction

A coloring of a simple graph G is to assign a color to each vertex of G so that adjacent vertices have different colors. The chromatic number of G, denoted by $\chi(G)$, is the smallest number of colors we need to color G. To compute the chromatic numbers of graphs is called the *graph coloring problem*, which has been researched for a long time in graph theory.

The first application of homotopy theory to this problem is Lovász's proof of Kneser's conjecture (see Section 2). He assigned a simplicial complex, called a neighborhood complex, to a graph and showed that the connectivity of the complex gives a lower bound for the chromatic number.

After that, several graph coloring complexes have been introduced by different authors. Box complex is one of them. It is a $\mathbb{Z}_2$-poset $B(G)$ assigned to a graph. Here we regard a poset as a topological space by its classifying space.

The purpose of this paper is a brief explanation of author's paper [14] and its background. Roughly speaking, the main result of [14] is to mention that the Kronecker double covering over G completely determines the (non-equivariant) poset structure of the box complex. The Kronecker double covering over G is the graph $K_2 \times G$. Here K_2 is the graph consisting of two vertices and one edge connecting them, and the notation "$\times$" means the categorical product. See Section 4 for details.

Section 2 and Section 3 are devoted to the background of neighborhood complexes and box complexes. For a more concrete introduction to this subject, we refer to [12] or [10].

In Section 5, we mention the statement of the result. As an application of it, we can construct graphs whose chromatic numbers are different but whose box complexes are (non-equivariantly) isomorphic.

We closed this section by giving precise definitions relating to the graph coloring problem. A graph is a pair $G = (V(G), E(G))$ consisting of a set $V(G)$ with a symmetric subset $E(G)$ of $V(G) \times V(G)$. Namely, a pair (v, w) of vertices of G is contained in

$E(G)$ if and only if its transpose (w, v) is contained in $E(G)$. Therefore our graphs are non-directed, may have loops, but have no multiple edges. A graph homomorphism is a map $f : V(G) \to V(H)$ with $(f \times f)(E(G)) \subset E(H)$. The complete graph K_n is defined by $V(K_n) = \{0, 1, \cdots, n-1\}$ and $E(K_n) = \{(x, y) \mid x \neq y\}$. Then the chromatic number $\chi(G)$ is formulated as the number

$$\chi(G) = \inf\{n \geq 0 \mid \text{There is a graph homomorphism from } G \text{ to } K_n.\}.$$

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2 Neighborhood complex

In this section, we shall review the definition and known results concerning with neighborhood complexes.

Let G be a graph and v a vertex of G. The neighbor $N(v)$ of G is the set of vertices of G adjacent to v. Note that if v is not looped then $N(v)$ does not contain v. The neighborhood complex $N(G)$, introduced in Lovász [11], is the abstract simplicial complex defined as follows:

- The underlying set of $N(G)$ is $V(G)$.
- A finite subset $\sigma \subset V(G)$ is a simplex if there is a vertex $v \in V(G)$ with $\sigma \subset N(v)$.

For example, the neighborhood complex of the complete graph K_n is the boundary of $(n-1)$-simplex, and hence is homeomorphic to S^{n-2}. Let C_n denote the n-cycle graph for $n \geq 3$. If n is odd then $N(C_n)$ is the 1-sphere. $N(C_4)$ is homotopy equivalent to S^0, and $N(C_{2n})$ with $n \geq 2$ is homotoopy equivalent to $S^1 \sqcup S^1$.

For a space X, we write $\mathrm{conn}(X)$ to indicate the largest integer $n \geq -1$ such that X is n-connected. Here (-1)-connectivity means non-emptiness. If X is contractible then we regard $\mathrm{conn}(X) = +\infty$, and if X is the empty space then $\mathrm{conn}(X) = -\infty$.

Theorem 2.1 (Lovász, 1978). *If a graph G is n-connected then $\chi(G)$ is greater than $n+2$. Namely, we have $\chi(G) > \mathrm{conn}(N(G)) + 2$.*

This is a consequence of Borsuk-Ulam's theorem. The outline of the proof used box complex will be given in Section 3. Lovász used Theorem 2.1 to prove Kneser's conjecture.

Kneser's conjecture [9] asserts that if the set of k-subsets of the n-point set $X = \{0, 1, \cdots, n-1\}$ is divided into $(n-2k+1)$-classes, then there are two disjoint subsets of X contained in the same class. This conjecture was translated into the following graph coloring problem: For a pair of positive integers n, k with $n \geq 2k$, Kneser's graph $KG_{n,k}$ is the graph defined by

$$V(KG_{n,k}) = \{\sigma \subset \{0, 1, \cdots, n-1\} \mid \#\sigma = k\},$$

$$E(KG_{n,k}) = \{(\sigma, \tau) \mid \sigma \cap \tau = \emptyset\}.$$

Then Kneser's conjecture is equivalent to $\chi(KG_{n,k}) = n - 2k + 2$. (Precisely, the assertion of Kneser's conjecture is equivalent to $\chi(KG_{n,k}) \geq n - 2k + 2$. However, it is easy to show $\chi(KG_{n,k}) \leq n - 2k + 2$. In fact one can construct an $(n - 2k + 2)$-coloring of $KG_{n,k}$ by induction on n. First, since $KG_{2k,k}$ is a disjoint union of K_2's, we have that $\chi(KG_{2k,k}) = 2$. Suppose that there is a coloring $f : KG_{n,k} \to K_{n-2k+2}$. Then we have a coloring $g : KG_{n+1,k} \to K_{n-2k+3}$ defined by

$$g(\sigma) = \begin{cases} f(\sigma) & (n+1 \notin \sigma) \\ n - 2k + 3 & (n+1 \in \sigma). \end{cases}$$

This completes the proof of $\chi(KG_{n,k}) \leq n - 2k + 2$.)

Lovász showed the following theorem.

Theorem 2.2 (Lovász [11], 1978). *The neighborhood complex of $KG_{n,k}$ is $(n - 2k - 1)$-connected.*

Combining Theorem 2.1 and Theorem 2.2, Lovász deduced Kneser's conjecture $\chi(KG_{n,k}) = n - 2k + 2$.

As a related topic, we mention stable Kneser's graphs. A subset S of $\mathbb{Z}/n$ is *stable* if $x \in S$ implies $x + 1 \notin S$. Stable Kneser's graph $SG_{n,k}$ is defined as follows: The vertices of $SG_{n,k}$ are stable k-subsets of $\mathbb{Z}/n$, and two stable k-subsets are adjacent if they are disjoint. Clearly stable Kneser's graph $SG_{n,k}$ is a subgraph of Kneser's graph $KG_{n,k}$ and hence $\chi(SG_{n,k}) \leq \chi(KG_{n,k})$. Soon after Lovász [11], the following theorem stronger than Kneser's conjecture was proven by Schrijver [15]. Here a graph G is called *vertex critical* if every subgraph H of G with $V(H) \subsetneq V(G)$ satisfies $\chi(H) < \chi(G)$.

Theorem 2.3 (Schrijver [15], 1978). *Stable Kneser's graph $SG_{n,k}$ is vertex critical and $\chi(SG_{n,k}) = n - 2k + 2$.*

The homotopy types of neighborhood complexes of stable Kneser's graphs were determined:

Theorem 2.4 (Björner-Longueville [2] 2003). *The neighborhood complex of $SG_{n,k}$ is homotopy equivalent to the $(n - 2k)$-sphere.*

3 Box complex

A partially ordered set is called a poset, for short. The order complex $\Delta(P)$ of P is the simplicial complex whose simplices are finite chains in P. It is known that the classifying space of P is naturally homeomorphic to the geometric realization of $\Delta(P)$. In this article, the classifying space of a poset P is denoted by $|P|$. For a simplicial complex K, we write FK to mean the face poset of K. Then the triangulation determined by $|FK|$ is the barycentric subdivision of $|K|$ and hence they are homeomorphic.

The box complex of a graph is a $\mathbb{Z}_2$-space assigned to a graph. There are similar constructions as follows. For more detailed research, refer to [5], [13], or [18].

(1) Let G be a finite graph. Consider the antitone map $\nu : FN(G) \to FN(G)$ defined by $\nu(\sigma) = \{v \in V(G) \mid \sigma \subset N(v)\}$. (In the definition of ν, we use the assumption that G is finite.) The Lovász complex $L(G)$ is the induced subset of $FN(G)$ consisting of simplices $\sigma \in N(G)$ with $\nu^2(\sigma) = \sigma$. Then the restriction of ν to $L(G)$ determines the involution of $L(G)$. This complex was introduced by Lovász [11] in the proof of Kneser's conjecture, and detailed research on it is found in Walker [17].

    For a graph homomorphism $f : G \to H$, we have a $\mathbb{Z}_2$-map $L(f) : L(G) \to L(H)$ defined by $L(f)(\sigma) = \nu^2(f(\sigma))$ (note that $\nu^3 = \nu$). For a composable graph homomorphisms g and f, one can see $L(g) \circ L(f) \simeq_{\mathbb{Z}_2} L(g \circ f)$. However, $L(g) \circ L(f) \neq L(g \circ f)$ in general.

(2) Let G be a graph. The complex $B(G)$ is the subcomplex of $N(G) * N(G)$ whose simplices are $\sigma * \tau$ for simplices $\sigma, \tau \in N(G)$ with $\sigma \times \tau \subset E(G)$. The involution of $B(G)$ is obviously defined. A graph homomorphism $f : G \to H$ gives rise to a $\mathbb{Z}_2$-map $B(f) : B(G) \to B(H)$ and it satisfies $B(g \circ f) = B(g) \circ B(f)$.

(3) Let G be a graph. The complex $\mathrm{Bip}(G)$ is the poset

$$\{(\sigma, \tau) \mid \sigma \text{ and } \tau \text{ are non-empty subsets of } G \text{ with } \sigma \times \tau \subset E(G).\}$$

    ordered by $(\sigma, \tau) \leq (\sigma', \tau') \Leftrightarrow \sigma \subset \sigma'$ and $\tau \subset \tau'$. The involution of $\mathrm{Bip}(G)$ is defined by the correspondence $(\sigma, \tau) \leftrightarrow (\tau, \sigma)$, and a graph homomorphism $f : G \to H$ induces a $\mathbb{Z}_2$-map $\mathrm{Bip}(f) : \mathrm{Bip}(G) \to \mathrm{Bip}(H)$, $(\sigma, \tau) \mapsto (f(\sigma), f(\tau))$. This complex is isomorphic to the Hom complex $\mathrm{Hom}(K_2, G)$, which was investigated in Babson-Kozlov [1].

In the reference, authors usually called the complex $B(G)$ the box complex.[1]

[1] In Section 5 we call $\mathrm{Bip}(G)$ the box complex of G since there is no appropriate term indicating the complex $\mathrm{Bip}(G)$.

Theorem 3.1 (Csorba et al [5], Živaljević [18]). *The above constructions of $\mathbb{Z}_2$-spaces are naturally $\mathbb{Z}_2$-homotopy equivalent. Moreover, they are naturally homotopy equivalent to neighborhood complexes*

A fundamental result of the box complex is the following theorem. Here we consider the k-sphere S^k as a $\mathbb{Z}_2$-space by its antipodal map. For its proof, we refer to [1], [12], or [10] for example.

Theorem 3.2. *The box complex of the complete graph K_n is $\mathbb{Z}_2$-homotopy equivalent to the $(n-1)$-sphere S^{n-2} for $n \geq 1$.*

For a $\mathbb{Z}_2$-space X, set

$$\mathrm{ind}(X) = \inf\{k \geq -1 \mid \text{There is a } \mathbb{Z}_2\text{-map from } X \text{ to } S^k\}.$$

Note that if there is a graph homomorphism from G to K_n then there is a $\mathbb{Z}_2$-map from $B(G)$ to $B(K_n) = S^{n-2}$. Therefore we have the following.

Corollary 3.3. $\chi(G) \geq \mathrm{ind}(B(G)) + 2$.

To deduce Theorem 2.1, it suffices to show $\mathrm{conn}(X) + 1 \leq \mathrm{ind}(X)$. Note that if the $\mathbb{Z}_2$-space X is k-connected then there is a $\mathbb{Z}_2$-map from S^{k+1} to X. Suppose $\mathrm{ind}(X) = m$ and let $X \to S^m$ be a $\mathbb{Z}_2$-map. Then there is a $\mathbb{Z}_2$-map from $S^{k+1} \to S^m$. Then Borsuk-Ulam's theorem implies $k + 1 \leq m$. Therefore we have $\mathrm{conn}(X) + 1 \leq \mathrm{ind}(X)$.

The difference between the inequality of Corollary 3.3 can be arbitrarily bad. In fact Walker [17] showed that for a positive integer n, there is a finite graph G whose box complex is $\mathbb{Z}_2$-homotopy equivalent to a 1-dimensional $\mathbb{Z}_2$-CW-complex (and hence $\mathrm{ind}(B(G)) \leq 1$) but $\chi(G) \geq n$.

By the definition of $SG_{n,k}$ (see Section 2), the dihedral group D_{2n} with order $2n$ acts on $SG_{n,k}$ in an obvious way, and Braun [3] showed that the automorphism group of $SG_{n,k}$ coincides with D_{2n}. Theorem 2.4 and Theorem 3.1 imply that $B(SG_{n,k})$ is homotopy equivalent to S^{n-2k}. Therefore $B(SG_{n,k})$ is a $\mathbb{Z}_2 \times D_{2n}$-space, and its topology was investigated by Schultz [16].

4 Kronecker double covering

Let G, H be graphs. The (Kronecker) product $G \times H$ is the graph defined by

$$V(G \times H) = V(G) \times V(H),$$

$$E(G \times H) = \{((x,y),(x',y')) \mid (x,x') \in E(G),\ (y,y') \in E(H)\}.$$

Then one can show that $G \times H$ is the categorical product in the category of graphs.

A graph homomorphism $p : G \to H$ is a covering if $p|_{N(v)} : N(v) \to N(p(v))$ is bijective. The Kronecker double covering is the second projection $K_2 \times G \to G$, $(a, v) \mapsto v$.

Here we give a brief review of the theory of Kronecker double coverings. Perhaps, the following formulation using 2-colored graphs might be first appeared in [14], but was essentially obtained by other authors, see [8] for example.

A 2-colored graph is a pair (X, ε) consisting of a graph X with a 2-coloring $\varepsilon : X \to K_2$. A graph homomorphism $f : X \to Y$ between 2-colored graphs are 2-colored if $\varepsilon_Y \circ f = \varepsilon_X$. We write $\mathcal{G}_{/K_2}$ to indicate the category of 2-colored graphs whose morphisms are 2-colored homomorphisms.

An odd involution of a 2-colored graph (X, ε) is a graph homomorphism $\tau : X \to X$ such that $\tau^2 = \mathrm{id}$ and $\varepsilon(\tau(x)) \neq \varepsilon(x)$ for all $x \in V(X)$. Define the category $\mathcal{G}^{odd}_{/K_2}$ as follows:

- An object of $\mathcal{G}^{odd}_{/K_2}$ is a triple (X, ε, τ) such that τ is an odd involution of a 2-colored graph (X, ε).
- A morphism from $(X, \varepsilon_X, \tau_X)$ to $(Y, \varepsilon_Y, \tau_Y)$ is a 2-colored homomorphism $f : X \to Y$ with $\tau_Y \circ f = f \circ \tau_X$.

Let $\mathcal{G}$ denote the category of graphs. Let G be a graph. The Kronecker double covering $K_2 \times G$ over G is regarded as an object of $\mathcal{G}^{odd}_{/K_2}$ as follows: The 2-coloring of $K_2 \times G$ is the first projection $K_2 \times G \to K_2$, $(a, v) \mapsto a$. The odd involution of $K_2 \times G$ is $(0, v) \leftrightarrow (1, v)$. Thus Kronecker double covering gives a functor from the category $\mathcal{G}$ of graphs to $\mathcal{G}^{odd}_{/K_2}$.

Lemma 4.1. *The functor*

$$K_2 \times - : \mathcal{G} \to \mathcal{G}^{odd}_{/K_2}$$

is an equivalence of categories.

Note that an involution τ of a graph G is identified with a $\mathbb{Z}_2$-action on G. We write G/τ to indicate the orbit space with respect to this action. The quasi-inverse of $K_2 \times - : \mathcal{G} \to \mathcal{G}^{odd}_{/K_2}$ is the quotient $(X, \tau, \varepsilon) \mapsto X/\tau$.

5 Results

In this section we assume that the term "box complex" means the $\mathbb{Z}_2$-poset $\mathrm{Bip}(G)$ introduced in (3) in the beginning of Section 3, and *we write $B(G)$ instead of* $\mathrm{Bip}(G)$. A principal result of [14] is the following. Let X be a 2-colored graph.

Theorem 5.1 (M. [14]). *Let G, H be graphs having no isolated vertices. Then the following hold.*

(1) The box complexes $B(G)$ and $B(H)$ are isomorphic as posets if and only if their Kronecker double coverings $K_2 \times G$ and $K_2 \times H$ are isomorphic as graphs.

(2) The box complexes $B(G)$ and $B(H)$ are isomorphic as $\mathbb{Z}_2$-posets if and only if they are isomorphic.

Here we give the "if" part of Theorem 4.2.(1). Define the *box complex for 2-colored graphs* to be the induced subposet

$$B_{/K_2}(X) = \{(\sigma, \sigma') \in B(G) \mid \sigma \subset \varepsilon_X^{-1}(0) \text{ and } \sigma' \subset \varepsilon_X^{-1}(1)\}$$

of the usual box complex $B(X)$. The box complex $B_{/K_2}$ for 2-colored graphs gives a functor from the category $\mathcal{G}_{/K_2}$ of 2-colored graphs to the category $\mathcal{P}$ of posets, and it is easy to show that $B(G) \cong B_{/K_2}(K_2 \times G)$ as posets. Therefore if $K_2 \times G \cong K_2 \times H$ then

$$B(G) \cong B(K_2 \times G) \cong B(K_2 \times H) \cong B(H)$$

and hence we have the "if" part of Theorem 4.2.(1).

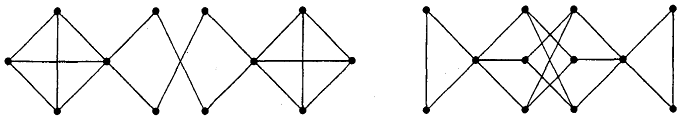
The "only if" part is due to the following lemma whose proof is a little complicated.

Lemma 5.2. *Suppose that X and Y are 2-colored graphs having no isolated vertices. Let $f : B_{/K_2}(X) \to B_{/K_2}(Y)$ be an isomorphism of posets. Then there is a unique 2-colored isomorphism $\hat{f} : X \to Y$ which induces f.*

Then Theorem 4.2.(2) is obtained as follows. Let G and H be graphs. It is clear that $G \cong H$ implies $B(G) \cong B(H)$ as $\mathbb{Z}_2$-posets. On the other hand, suppose that $f : B(G) \to B(H)$ is an isomorphism of $\mathbb{Z}_2$-posets. Since $B_{/K_2}(K_2 \times G) \cong B_{/K_2}(K_2 \times H)$, we have an isomorphism $\hat{f} : K_2 \times G \to K_2 \times H$ of 2-colored graphs which induces f. Then $\hat{f}$ commutes with their odd involutions, so we have that $K_2 \times G \cong K_2 \times H$ as objects in $\mathcal{G}^{odd}_{/K_2}$. Therefore Theorem 4.2.(2) follows.

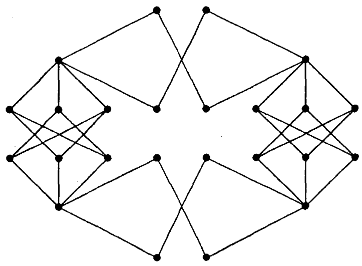
As an application of Theorem 5.1, we can construct graphs G, H with $B(G) \cong B(H)$ as posets but $\chi(G) \neq \chi(H)$.

Example 5.3. Consider the two graphs G and H depicted in Figure 1. Then their Kronecker double coverings $K_2 \times G$ and $K_2 \times H$ are isomorphic to the bipartite graph X depicted in Figure 2.



The graph G. The graph H.

Figure 1.



The graph X.

Figure 2.

To see this, consider two involutions τ_h and τ_v of X defined as follows. τ_h is the reflection in the central horizontal line of Figure 2, and τ_h is the reflection in the central vertical line of Figure 2. Then they are odd involutions of X and the quotient is $X/\tau_h = G$ and $X/\tau_v = H$. Therefore we have $K_2 \times G \cong X \cong K_2 \times H$.

It is easy to see that $\chi(G) = 4$ and $\chi(H) = 3$. Therefore we have $B(G) \cong B(H)$ as posets but $\chi(G) \neq \chi(H)$. Note that X consists of two $K_2 \times K_3$ and two $K_2 \times K_4$. Generalizing this construction, we have that for a pair (n, m) of integers greater than 1, there are graphs G, H such that $K_2 \times G \cong K_2 \times H$ (and hence $B(G) \cong B(H)$ as posets) but $\chi(G) = n$ and $\chi(H) = m$.

Therefore to determine the chromatic number from $B(G)$, we need to consider the equivariant topology of $B(G)$. On the other hand, Theorem 4.1.(2) implies that if $B(G) \cong B(H)$ as $\mathbb{Z}_2$-posets then $G \cong H$ and hence $\chi(G) = \chi(H)$. Thus we consider the following problem:

Question 5.4. *Is there a $\mathbb{Z}_2$-topological invariant of a box complex equivalent to the chromatic number $\chi(G)$?*

If we replace "$\mathbb{Z}_2$-topological invariant" to "$\mathbb{Z}_2$-homotopy invariant" in the above, then the following example given by Walker [17] is a counterexample.

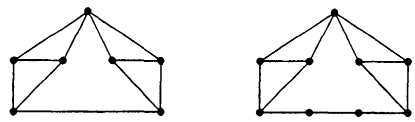


Figure 3.

In fact $\chi(G_1) = 3$ and $\chi(G_2) = 4$, but their Lovász complexes (see (1) in the beginning of Section 3) are $\mathbb{Z}_2$-homeomorphic.

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